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A thin spray model with collisions: existence and uniqueness of local smooth solutions

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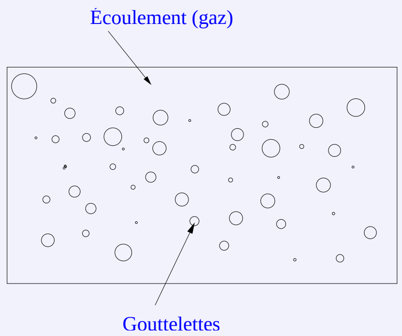

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Plan

I **Spray models**

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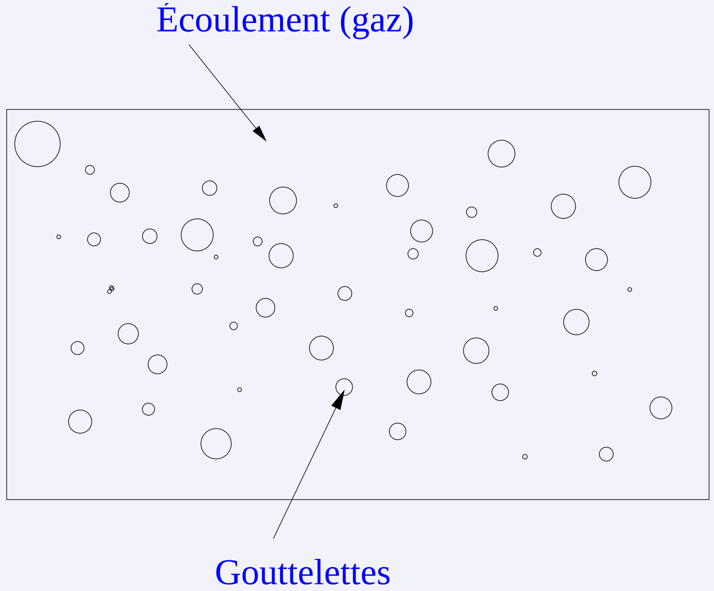
II **Existing results, main result and proof of the theorem**

I Spray models

Sprays = gaz with droplets



- P. O'Rourke. *Collective drop effects on vaporizing liquid sprays*. Thèse, Princeton University, 1981.
- F.A. Williams. *Combustion theory*. Benjamin Cummings, 1985.

Sprays : the continuous phase

- Continuous phase : described through compressible fluid mechanics

Macroscopic quantities (depending on (t, x))

- ⋆ density $\rho_g(t, x)$
- ⋆ velocity $u_g(t, x)$
- ⋆ total energy $E_g(t, x)$
- ⋆ pressure $p(t, x)$

Euler equations

$$
\begin{aligned}
\partial_t \rho_g + \nabla_x \cdot (\rho_g\, u_g) &= 0, \\
\partial_t (\rho_g\, u_g) + \nabla_x \cdot (\rho_g\, u_g \otimes u_g) + \nabla_x p &= 0, \\
\partial_t (\rho_g E_g) + \nabla_x \cdot ((\rho_g\, E_g + p) u_g) &= 0, \\
p_g = P(\rho_g, e_g) & \quad , \\
E_g = e_g + \frac{1}{2} {u_g}^2 & \quad .
\end{aligned}
$$

Sprays : dispersed phase

- Dispersed phase (droplets) : described by Vlasov-Boltzmann equation

  Probability Density Function $f(t, x, v, e...)$ where x position, v velocity, e internal energy...

  Vlasov-Boltzmann equation

$$\partial_t f + \underbrace{\nabla_x \cdot (f v)}_{\text{transport (advection)}} + \underbrace{\nabla_v \cdot (f \Gamma) + \nabla_e \cdot (f \phi)}_{\text{acceleration, energy transfer}} = \underbrace{Q(f, f)}_{\text{collision kernel}}$$

- More physical parameters can be added (like radii).

Sprays : description

- Spray= coupling between the two phases
- Sprays classification

$$\begin{aligned}
\partial_t(\rho_g) + \nabla_x \cdot (\rho_g u_g) &= 0, \\
\partial_t(\rho_g u_g) + \nabla_x \cdot (\rho_g u_g \otimes u_g) + \nabla_x p &= \int_{u_p, e_p} -m_p \Gamma f, \\
\partial_t(\rho_g E_g) + \nabla_x \cdot \left(\rho_g \left(E_g + \frac{p}{\rho_g}\right)\right) &= \iint_{v,e} -m_p f \Gamma \cdot v - m_p f \phi, \\
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f \Gamma) + \nabla_e \cdot (f \phi) &= Q(f, f),
\end{aligned}$$

- Very thin sprays

$$m_p \Gamma = D_p(u_g - v)$$

$$m_p \phi = 4\pi r_p \lambda Nu(T_g - T_p)$$

$$Q(f, f) = 0$$

- Thin sprays with collisions

$$m_p \Gamma = D_p(u_g - v)$$

$$m_p \phi = 4\pi r_p \lambda Nu(T_g - T_p)$$

II Existing results, main result and proof of the theorem

Spray:

- K. Domelevo and J-M. Roquejoffre. Existence and stability of travelling wave solutions in a kinetic model of two-phase flows. *Comm. Partial Differential Equations*, 24(1-2):61–108, 1999.

  Coupling between Burgers equation and Vlasov equation.

- C. Baranger and L. Desvillettes. Coupling Euler and Vlasov equations in the context of sprays : local smooth solutions. *J. Hyperbolic Differ. Equ.*, 3:1–26, 2006.

  Coupling between isentropic Euler equations and Vlasov equation.

System of equations

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g u_g) = 0\,,$$

$$\partial_t u_g + (u_g \cdot \nabla_x) u_g + \frac{\nabla_x p}{\rho_g} = -\frac{1}{\rho_g} \iint\limits_{v,e} F f dv de\,,$$

$$\partial_t e_g + u_g \cdot \nabla_x e_g + \frac{p}{\rho_g} \nabla_x \cdot u_g = \frac{1}{\rho_g} \iint\limits_{v,e} (F \cdot (u_g - v) - \phi)\, f dv de$$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (fF) + \partial_e (f\phi) = Q(f,f)\,,$$

$$F = -(v - u_g)\,,$$

$$\phi = -(e - e_g)\,,$$

$$p = (\gamma - 1)\rho_g e_g\,.$$

Collision kernel $Q(f,f)$

$$Q(f,f)(t,x,v,e) = \iiint_{\sigma\in\mathbb{S}^2, v_*\in\mathbb{R}^3, e_*\in\mathbb{R}^+} \left(f'_* f' - f_* f\right) |v - v^*| d\sigma dv_* de_*$$

avec,

- σ: $\sigma \in S^2$
- $v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma$: pre-collisionnal velocity
- ${v'}_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma$: pre-collisionnal velocity
- $e' = e$: pre-collisionnal internal energy
- ${e'}_* = e_*$: pre-collisionnal internal energy

Problem at "fixed gas"

Let' s take $g := f \exp(v^2 + e)$. g satisfies the following equation,

$$\partial_t g + v \cdot \nabla_x g + \nabla_v \cdot (gF) + \partial_e(g\phi) - 2v \cdot gF - g\phi = \Gamma(g, g) \qquad (1)$$

$$F = -(v - u_g)\,,$$

$$\phi = -(e - e_g),$$

where $\tilde{\rho}_g (= \rho_g - 1), u_g, \tilde{e}_g (= e_g - 1) \in \mathcal{C}([0,T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3))$,

and with the new collision kernel Γ defined as

$$\Gamma[g_1, g_2](t,x,v,e) = \Gamma^+[g_1,g_2](t,x,v,e) - \Gamma^-[g_1,g_2](t,x,v,e)$$

$$= \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-({v_*}^2 + e_*)) g_1(t,x,v'_*,e'_*) g_2(t,x,v',e') dv_* de_* d\sigma$$

$$-g_2(t,x,v,e) \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-({v_*}^2 + e_*)) g_1(t,x,v_*,e_*) dv_* de_* d\sigma.$$

Functional space of the study :

L^2 spaces and Sobolev spaces H^s with weights ($s \geq 5$),

$$||g|| = \left(\iiint_{x,v,e} g^2 dedvdx \right)^{\frac{1}{2}},$$

$$||g||_w = \left(\iiint_{x,v,e} g^2 wdedvdx \right)^{\frac{1}{2}},$$

où $w = 1 + v^2$.

Functions with finite energy (using Guo's ideas):

$$\mathcal{E}_{s,\varepsilon}(g(t,.,.,.)) = \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left[\frac{1}{2} \left|\left| \partial_\alpha^{\beta,\gamma} g \right|\right|^2 (t) + \int_0^t 2(1-\varepsilon) \left|\left| \partial_\alpha^{\beta,\gamma} g(u) \right|\right|_w^2 du \right].$$

Derivative terms control

Transport term:

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left| \iiint \partial_\alpha^{\beta,\gamma} g \, \partial_\alpha^{\beta,\gamma} \left(\nabla_x \cdot (gv)\right) dvdedx \right| \leq C||g||_{H^s}^2 .$$

Thermic term:

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left| \iiint \partial_\alpha^{\beta,\gamma} g \, \partial_\alpha^{\beta,\gamma} \nabla_v \cdot (gF) dvdedx \right| \quad \leq \quad C(||g||_{H^s}^2 ||u_g||_{H^s} + ||g||_{H^s}^2).$$

Non linear terms

Collision term:

$$\left| \iiint \partial_\alpha^{\beta,\gamma} \left(\Gamma(g_1, g_2)\right) \partial_\alpha^{\beta,\gamma} g_3 dv de dx \right| \leq C \left[\sum_{|\delta| \leq 3} \left|\left| \partial_{\alpha_2+\delta}^{\beta_2,\gamma_2} g_2 \right|\right| \right] \left|\left| \partial_{\alpha_1}^{\beta_1,\gamma_1} g_1 \right|\right| \left|\left| \partial_\alpha^{\beta,\gamma} g_3 \right|\right|_w + C \left[\sum_{|\delta| \leq 3} \left|\left| \partial_{\alpha_1+\delta}^{\beta_1,\gamma_1} g_1 \right|\right| \right] \left|\left| \partial_{\alpha_2}^{\beta_2,\gamma_2} g_2 \right|\right| \left|\left| \partial_\alpha^{\beta,\gamma} g_3 \right|\right|_w .$$

Transport term:

$$\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \iiint \partial_\alpha^{\beta,\gamma} g \, \partial_\alpha^{\beta,\gamma} (gF \cdot v) dv de dx \leq -(1-\varepsilon) \sum_{|\alpha|+|\beta|+|\gamma| \leq s} ||\partial_\alpha^{\beta,\gamma} g||_w^2 + C_1 ||g||_{H^s}^2 + \frac{C_2}{\varepsilon} ||g||_{H^s}^2 ||u_g||_{H^s}^2 .$$

Thermic term:

$$\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left| \iiint \partial_\alpha^{\beta,\gamma} g \, \partial_\alpha^{\beta,\gamma} (g\phi) dv de dx \right| \leq C_1 ||g||_{H^s}^2 (1 + ||\tilde{e}_g||_{H^s}).$$

A priori estimate

Proposition 1
Let g be a nonnegative solution of (1) in $\mathcal{C}([0,T],H^s(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+))\cap\mathcal{C}^1([0,T],H^{s-1}(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+))$, *with* ρ_g *and* e_g *strictly positive functions and* $\tilde\rho_g=\rho_g-1$, u_g *and* $\tilde e_g=e_g-1$ *belonging to* $\mathcal{C}([0,T],H^s(\mathbb{R}^3))\cap\mathcal{C}^1([0,T],H^{s-1}(\mathbb{R}^3))$ *for some* $T>0$. *For all* $s\geq 5$, $1>\varepsilon>0$, *we have the following a priori estimate for the energy* $\mathcal{E}_{s,\varepsilon}$,

$$\frac{d}{dt}\mathcal{E}_{s,\varepsilon}\leq\frac{C}{\varepsilon}\left(1+||u_g||^2_{H^s}+||\tilde e_g||_{H^s}+\mathcal{E}_{s,\varepsilon}\right)\mathcal{E}_{s,\varepsilon}. \tag{2}$$

where C *depends only on the physical constants and* s.

Solving the problem at fixed gas

Iterative scheme

$$\partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + (u_g - v) \cdot \nabla_v g^{n+1} + (e_g - e)\partial_e g^{n+1} = \Gamma^+(g^n, g^n) + h(g^n) g^{n+1},$$

$h(g_n)$ défini par:

$$\begin{aligned} h(g_n)(t,x,v,e) &= 4 + 2v \cdot (u_g - v) + (e_g - e) \\ &- \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-({v_*}^2 + e_*)) g_n(t,x,v_*,e_*) dv_* de_* d\sigma. \end{aligned}$$

$$g(0,.,.,.) = g_0(.,.,.).$$

Characteristics:

$$\begin{aligned} \frac{d}{dt}X &= V &, \quad X(s;x,v,e,s) = x, \\ \frac{d}{dt}V &= -(V - u_g(t,X)) &, \quad V(s;x,v,e,s) = v, \\ \frac{d}{dt}E &= -(E - e_g(t,X)) &, \quad E(s;x,v,e,s) = e. \end{aligned}$$

Theorem at fixed gas

We suppose that ρ_g and e_g are strictly positive functions. Moreover, $\tilde{\rho}_g = \rho_g - 1$, u_g et $\tilde{e}_g = e_g - 1$ belong to $\mathcal{C}([0,T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3))$ for a given $T > 0$ and s is an integer such that $s \geq 5$. We suppose that $||g_0||_{H^s} < +\infty$. Then :

1 There exists $T' > 0$ such that there is a solution g to the equation

$$\partial_t g + v \cdot \nabla_x g + \nabla_v \cdot (gF) + \partial_e (g\phi) - 2v \cdot gF - g\phi = \Gamma(g,g)$$
$$F = -(v - u_g), \phi = -(e - e_g),$$

with g_0 for initial data. Besides, g remains positive and

$$\mathcal{E}_{s,\frac{1}{2}}(g(t)) = \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left[\frac{1}{2} \left|\left|\partial_\alpha^{\beta,\gamma} g\right|\right|^2 (t) + \int_0^t \left|\left|\partial_\alpha^{\beta,\gamma} g\right|\right|_w^2 (u) du \right]$$

remains bounded on $[0, T']$. Finally, T' is controlled through

$$T' \geq \frac{C}{\left[1 + \sup_{0 \leq t \leq T} ||u_g||_{H^s}^2 + \sup_{0 \leq t \leq T} ||\tilde{e}_g||_{H^s} + \frac{3}{2}||g_0||_{H^s}^2\right]}$$

where C is strictly positive and depend on s. Eventually, $g \in \mathcal{C}([0,T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$.

Theorem at fixed gas

2 If g_1 et g_2 are two positive solutions of equation (1) in $\mathcal{C}([0,T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ with same initial data, same data ρ_g, u_g et e_g, and satisfies $\sup_{0\leq t\leq T} \mathcal{E}_{s,\frac{1}{2}}(g_1(t,.)) < +\infty$ et $\sup_{0\leq t\leq T} \mathcal{E}_{s,\frac{1}{2}}(g_2(t,.)) < +\infty$ then $g_1 = g_2$.

Hyperbolic part of the system

$I =]0,+\infty[\times\mathbb{R}^3\times]0,+\infty[$, $s\in\mathbb{N}$ such that $s\geq 5$. I_1, I_2 open sets of I such that $\overline{I_1}\subset I_2$ et $\overline{I_1}$, $\overline{I_2}$ are compact in I.

We note $U_g = {}^t(\rho_g, u_g, e_g)$, $\delta_{j,i}$ the Kronecker's symbol, and c the sound speed of the gas defined by $c := \sqrt{\gamma(\gamma-1)e_g}$. The fluid part of the system is symmetrized thanks to the following formula:

$$S(U_g)\partial_t U_g + \sum_i (SA_i)(U_g)\partial_{x_i} U_g = S(U_g)\, b(U_g, f),$$

$$\text{where } S = \begin{pmatrix} \left(\frac{(\gamma-1)e_g}{\rho_g}\right)^2 & 0 & 0 \\ 0 & \frac{1}{\gamma}c^2 & 0 \\ 0 & 0 & (\gamma-1) \end{pmatrix},$$

$$A_i = u_{g\,i} Id_5 + \begin{pmatrix} 0 & \rho_g\delta_{1,i} & \rho_g\delta_{2,i} & \rho_g\delta_{3,i} & 0 \\ \frac{(\gamma-1)e_g}{\rho_g}\delta_{1,i} & 0 & 0 & 0 & e_g\delta_{1,i} \\ \frac{(\gamma-1)e_g}{\rho_g}\delta_{2,i} & 0 & 0 & 0 & e_g\delta_{2,i} \\ \frac{(\gamma-1)e_g}{\rho_g}\delta_{3,i} & 0 & 0 & 0 & e_g\delta_{3,i} \\ 0 & (\gamma-1)e_g\delta_{1,i} & (\gamma-1)e_g\delta_{2,i} & (\gamma-1)e_g\delta_{3,i} & 0 \end{pmatrix},$$

Hyperbolic part of the system (II)

$$b = \begin{pmatrix} 0 \\ -\int \frac{1}{\rho_g} f(u_{g1} - v) dvde \\ -\int \frac{1}{\rho_g} f(u_{g2} - v) dvde \\ -\int \frac{1}{\rho_g} f(u_{g3} - v) dvde \\ -\int \frac{1}{\rho_g} f(e_g - e) dvde + \int \frac{1}{\rho_g} f\,(u_g - v)^2\, dvde \end{pmatrix}.$$

For $s \geq 5$, if $f\,\exp(v^2 + e)(t,.)$ is in $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$, if moreover $(\rho_g - 1, u_g, e_g - 1)\,(t,.)$ is in $(H^s(\mathbb{R}^3))^5$ and belongs to I_2, then b is in $H^s(\mathbb{R}^3)$ and we have the following control:

$$\begin{aligned} ||b(t,.)||_{H^s(\mathbb{R}^3)} \quad &\leq \quad C(s, \overline{I_2}) \times (1 + ||\rho_g - 1||_{H^s}) \\ &\times ||f\,\exp(v^2 + e)(t,.)||_{H^s} \times \left(1 + ||e_g - 1||_{H^s} + ||u_g||^2_{H^s}\right), \end{aligned} \tag{3}$$

with C constant only depending on the compact set $\overline{I_2}$ and on the Sobolev index s.

Coupled system

$I =]0,+\infty[\times\mathbb{R}^3\times]0,+\infty[$, $s\in\mathbb{N}$ such that $s\geq 5$. I_1, I_2 open sets of I such that $\overline{I_1}\subset I_2$ et $\overline{I_1}$, $\overline{I_2}$ are compact in I. Let $(\rho_{g_0}, u_{g_0}, e_{g_0}) : \mathbb{R}^3 \to I_1$ be functions satisfying $\tilde{\rho}_{g_0} = \rho_{g_0} - 1 \in H^s(\mathbb{R}^3)$, $u_{g_0} \in H^s(\mathbb{R}^3)$ et $\tilde{e}_{g_0} = e_{g_0} - 1 \in H^s(\mathbb{R}^3)$. Let $\tilde{f}_0 = f_0 \exp(v^2+e) : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+ \to \mathbb{R}^+$ be a function in $H^s(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+)$.

- Then, one can find $T>0$ such that there exists a solution $(\rho_g, u_g, e_g; f)$ to the global system which belongs to $C^1([0,T]\times\mathbb{R}^3, \overline{I_2}) \times C^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+, \mathbb{R}^+)$. Furthermore, $\tilde{\rho}_g (= \rho_g - 1)$, u_g, $\tilde{e_g} (= e_g - 1) \in \mathcal{C}([0,T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3))$, $\tilde{f} = f\exp(v^2+e) \in \mathcal{C}([0,T], H^s(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+))$.

- Finally, if $(\tilde{\rho}_{g_1}, u_{g_1}, \tilde{e}_{g_1}; \tilde{f}_1)$ and $(\tilde{\rho}_{g_2}, u_{g_2}, \tilde{e}_{g_2}; \tilde{f}_2)$ belong to $(\mathcal{C}([0,T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3))) \times (\mathcal{C}([0,T], H^s(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+)))$, if $(\rho_{g_1}, u_{g_1}, e_{g_1}; f_1)$ and $(\rho_{g_2}, u_{g_2}, e_{g_2}; f_2)$ belong to $C^1([0,T]\times\mathbb{R}^3, \overline{I_2}) \times C^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+, \mathbb{R}^+)$ and if they satisfy the equations of the system, then:

$$\rho_{g_1} = \rho_{g_2}, u_{g_1} = u_{g_2}, e_{g_1} = e_{g_2} \text{ and } f_1 = f_2.$$

Conclusion and perspectives

- Solutions local in time
- What if f_0 is only in H^s ?
- What happens for thick sprays (when the volume fraction occupied by the particles is not negligible)?