Of the $k - \varepsilon$

incompressible Model

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Use of the model:

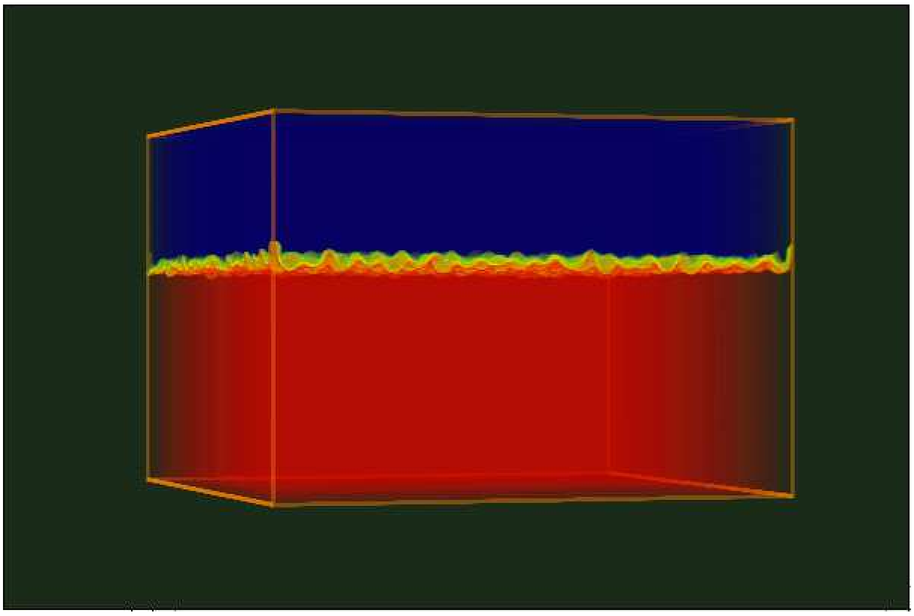
- description of turbulence

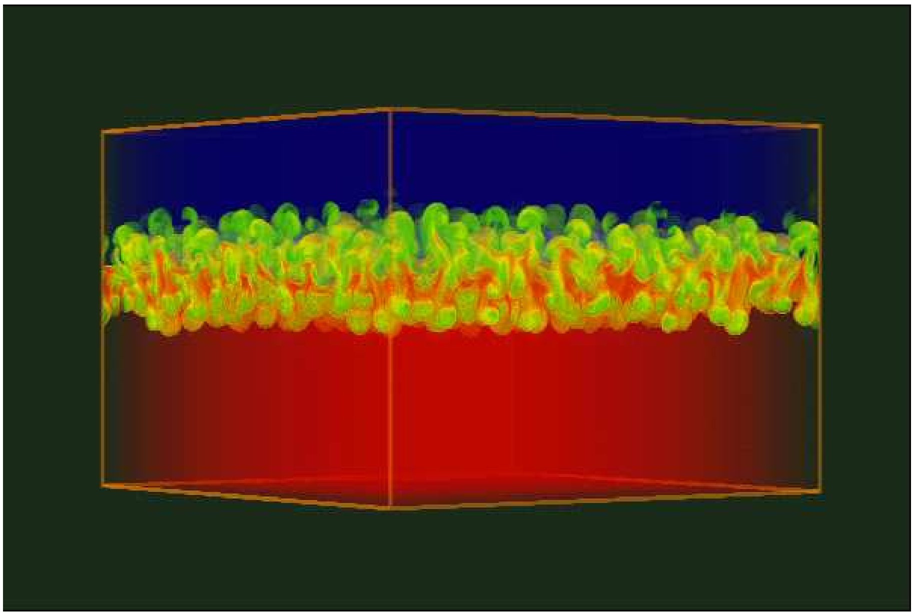
- first order statistical model

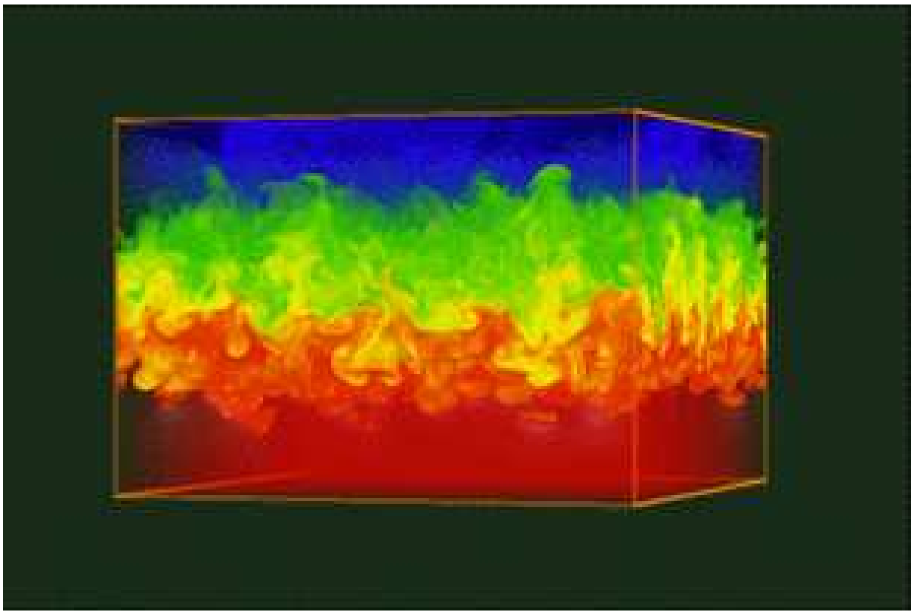
- Rayleigh Taylor instabilities compressible model)

- aeronautics

Rayleigh-Taylor simulations







Equations

$$\frac{\partial U}{\partial t} + U\nabla U + \nabla P - \nu\Delta U + \nabla \cdot R = 0$$

$$\nabla \cdot U = 0$$

$$\frac{\partial k}{\partial t} + U\nabla k - \frac{c_\mu}{2}\frac{k^2}{\varepsilon}|\nabla U + \nabla U^T|^2 - \nabla\cdot(c_\mu \frac{k^2}{\varepsilon}\nabla k) + \varepsilon = 0$$

$$\frac{\partial \varepsilon}{\partial t} + U\nabla \varepsilon - \frac{c_1}{2}\frac{k^2}{\varepsilon}|\nabla U + \nabla U^T|^2 - \nabla\cdot(c_\varepsilon \frac{k^2}{\varepsilon}\nabla \varepsilon) + c_2\frac{\varepsilon^2}{k} = 0$$

$c_1 = 0.126$
$c_2 = 1.92$
$c_\mu = 0.09$
$c_\varepsilon = 0.07$

$$R = \frac{2}{3}k\mathbf{I} - (\nu + c_\mu \frac{k^2}{\varepsilon})(\nabla U + \nabla U^T)$$

$\mathbf{R}$: Reynolds stress

U: mean flow

u flow

$u' = u - U$ fluctuating velocity

$\mathbf{I}$: identity matrix

ν: reduced viscosity

k: kinetic energy of the turbulence

ε : rate of dissipation of turbulent energy

Simple model

$$\frac{\partial k}{\partial t} - \nabla \cdot \left(c_\mu \frac{k^2}{\varepsilon} \nabla k \right) + \varepsilon = 0$$

$$\frac{\partial \varepsilon}{\partial t} - \nabla \cdot \left(c_\varepsilon \frac{k^2}{\varepsilon} \nabla \varepsilon \right) + c_2 \frac{\varepsilon^2}{k} = 0$$

$$c_2 = 1.92$$
$$c_\varepsilon = 0.07$$
$$c_\mu = 0.09$$

Non dimensional equations

$$\frac{\partial k}{\partial t} - \eta \nabla \cdot (\frac{k^2}{\varepsilon} \nabla k) + A\varepsilon = 0$$

$$\frac{\partial \varepsilon}{\partial t} - \eta \nabla \cdot \left(\frac{c_\varepsilon}{c_\mu} \frac{k^2}{\varepsilon} \nabla \varepsilon \right) + c_2 A \frac{\varepsilon^2}{k} = 0$$

In some dense plasma physics applications:

$$A = \frac{\varepsilon^0 T}{k^0} \simeq 1$$

$$\eta = c_\mu \frac{(k^0)^2 T}{\sigma_k \varepsilon^0 L^2} \ll 1$$

First results

$$\forall x \in I \text{ et } \forall t \in [0, T]$$

$$k(x,t) \leq k_{max}.$$

$$\varepsilon(x,t) \leq \varepsilon_{max}.$$

$$k(t,x) \geq k_{min} - A\varepsilon_{max} t$$

$$\varepsilon(t,x) \geq \frac{\varepsilon_{min}}{1 - c_2 \frac{\varepsilon_{min}}{\varepsilon_{max}} \ln(1 - \frac{tA\varepsilon_{max}}{k_{min}})}$$

Asymptotic analysis of $k - \varepsilon$ model

0-Order System:

$$\frac{\partial k_0}{\partial t} + A\varepsilon_0 = 0$$

$$\frac{\partial \varepsilon}{\partial t} + c_2 A \frac{{\varepsilon_0}^2}{k_0} = 0$$

$$k_0(0,.) = k^0(.)$$

$$\varepsilon_0(0,.) = \varepsilon^0(.)$$

First order system:

$$\frac{\partial k_1}{\partial t} - \nabla \cdot \left(\frac{{k_0}^2}{\varepsilon_0} \nabla k_0 \right) + A\varepsilon_1 = 0$$

$$\frac{\partial \varepsilon_1}{\partial t} - \nabla \cdot \left(\frac{\sigma_k}{\sigma_\varepsilon} \frac{{k_0}^2}{\varepsilon_0} \nabla \varepsilon_0 \right) + c_2 A \left(\frac{2\varepsilon_0 \varepsilon_1}{k_0} - \frac{{\varepsilon_0}^2 k_1}{{k_0}^2} \right) = 0$$

$$k_1(0,.) = 0$$

$$\varepsilon_1(0,.) = 0$$

First theorem

Theorem 1 *:*

If k^0 et ε^0 belong to $H^7(\mathbb{R}/\mathbb{Z})$, k et ε belong to $\mathcal{C}^1([0,T],H^3(\mathbb{R}/\mathbb{Z}))$ for $t \leq T < \frac{k_{min}}{A\varepsilon_{max}}$, there exist a function H(T) such as :

$$\|k - k_0 - \eta k_1\|_\infty(t) \leq \sqrt{CHt}\ \eta^{\frac{3}{2}}$$

$$\|\varepsilon - \varepsilon_0 - \eta\varepsilon_1\|_\infty(t) \leq \sqrt{CHt}\ \eta^{\frac{3}{2}}$$

$$\|\nabla(k - k_0 - \eta k_1)\|_\infty(t) \leq \sqrt{CHt}\ \eta^{\frac{3}{2}}$$

$$\|\nabla(\varepsilon - \varepsilon_0 - \eta\varepsilon_1 -)\|_\infty(t) \leq \sqrt{CHt}\ \eta^{\frac{3}{2}}$$

Result

k ,ε remain positive.

Global system- Maximum priciple

$$\frac{dk_{min}}{\partial t} \geq -C||\varepsilon||_{H^3}$$

$$\frac{dk_{max}}{\partial t} \leq \frac{2c_\mu C^2}{k_{min}}||U||^2_{H^3}||\varepsilon||^2_{H^3}$$

$$\frac{d\varepsilon_{min}}{\partial t} \geq -c_2 C^2 \frac{||\varepsilon||^2_{H^3}}{k_{min}}$$

$$\frac{d\varepsilon_{max}}{\partial t} \leq 2c_1 C^3 ||k||_{H^3}||U||^2_{H^3}$$

Global system,a priori estimate 1

Theorem 2 (H^3 estimate for U)
There exists a two-variable polynomial function Q such as:

$$\begin{aligned}\frac{d}{dt}||U||^2_{H^3} \leq\ & Q(k_{max}, \frac{1}{\varepsilon_{min}}) \\ & \times(1 + ||k||^2_{H^3})^3 \\ & \times||1 + ||\varepsilon||^2_{H^3})^3 \\ & \times(1 + ||U||_{H^3})^3\end{aligned}$$

The same kind of estimates hold for k and ε.

Global system, a priori estimate 2

Let's define f by:

$$
\begin{aligned}
f(t) \;=\; & ||k(t)||^2_{H^3} + ||\varepsilon(t)||^2_{H^3} + ||U(t)||^2_{H^3} \\
& + k_{max}(t) + \varepsilon_{max}(t) \\
& + \frac{1}{k_{min}(t)} + \frac{1}{\varepsilon_{min}(t)}
\end{aligned}
$$

f satisfies the following equation:

$$\frac{d}{dt} f \leq C(1 + f^6)$$

Conclusion:

k and ε remain strictly positive in small time.

The norms of U, k and ε do not explode.